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Odd Modular Edge Irregularity Strength of Graphs

K. Muthugurupackiam, K.S. Krishnamohan

Department of Mathematics
Rajah Serfoji Government College
Thanjavur, Tamilnadu, India
gurupackiam@yahoo.com, kskrishnamohan@gmail.com

S. Ramya
Department of Mathematics
Bharathidasan University Constituent College
Orathanadu, Tamilnadu, India
manoramya.8036@gmail.com

Abstract

In this paper, we introduce the concept of odd modular edge irregularity strength of graphs. We investigate exact value of odd modular irregularity strength for some families of graphs. Further we present a lower bound for this parameter.

Key words: irregular labeling, modular irregular labeling, odd modular edge irregular labeling, vertex k-labeling, irregularity strength, modular irregularity strength, odd modular edge irregularity strength.

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1 Introduction

Let $G = (V, E)$ be a simple graph. A map that carries vertex set (edge set or both) as domain to the positive integers {1,2,. . . , k} is called *vertex k-labeling* (*edge k-labeling or total k-labeling*). Chartrand et al. [4] introduced the problem of determining the irregularity strength of a graph. Consider a simple graph $G = (V, E)$, having at most one isolated vertex and no component of order 2, together with an edge k-labeling $\lambda : E(G) \to \{1, 2, \dots, k\}$. For each vertex v of G, define the weight $w_\lambda(v) = \sum_{u \in N(v)} \lambda(uv)$

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and λ is called irregular if for every pair of distinct vertices u and v of G, $w_\lambda(u) \neq w_\lambda(v)$. The minimum value of k, for which the graph G has an irregular k-labeling, is called irregularity strength of G and denoted by $s(G)$.

The parameter irregularity strength of a graph is attracted by numerous authors. Aigner and Triesh [1] proved that $\mathrm{s}(G) \leq n-1$ if G is a connected graph of order n, and $\mathrm{s}(G) \leq n+1$ otherwise. Nierhoff [13] refined their method and showed that $\mathrm{s}(G) \leq n-1$ for all graphs with finite irregularity strength, except for K_3. For star graphs this bound is tight. In particular Faudree and Lehel [5] showed that if G is d-regular ($d \geq 2$), then $\left\lceil \frac{n+d-1}{d} \right\rceil \leq \mathrm{s}(G) \leq \left\lceil \frac{n}{2} \right\rceil + 9$, and they conjectured that $\mathrm{s}(G) \leq \left\lceil \frac{n}{d} \right\rceil + c$ for some constant c. Przybylo in [14] proved that $\mathrm{s(G)} \leq 16\frac{\mathrm{n}}{\mathrm{d}} + 6$. Kalkowski, Karonski and Pfender [8] showed that $\mathrm{s}(G) \leq 6\frac{n}{\delta} + 6$, where δ is the minimum degree of graph G. Currently Majerski and Przybylo [9] proved that $\mathrm{s}(G) \leq (4 + o(1))\frac{n}{\delta} + 4$ for graphs with minimum degree $\delta \geq \sqrt{n} \ln n$. For recent survey of graph labeling refer [6].

Ahmad et al. [2] defined edge irregularity strength of a graph as follows: Consider a simple graph G together with a vertex k-labeling $\phi : V(G) \to \{1, 2, \ldots, k\}$. The weight of an edge $e = xy$ is the sum of the labels $\phi(x)$ and $\phi(y)$ and denoted by $wt(e)$. A vertex k-labeling is defined to be an edge irregular k-labeling of the graph G if for every two different edges f and g there is $wt(f) \neq wt(g)$. The minimum k for which the graph G has an edge irregular k-labeling is called the edge irregularity strength of G, and denoted by $es(G)$. The lower bound of $es(G)$ was given by the following inequality

$$es(G) \geq max\{\left\lceil \frac{E(G)+1}{2} \right\rceil, \Delta\}$$

where Δ is the maximum degree of graph G. Tarawneh et al. [7], determined the exact value of edge irregularity strength of corona graphs of path P_n with P_2, P_n with K_1 and P_n with S_m.

Martin Bača et al. [10] introduced modular irregularity strength of a graph. Let $G = (V(G), E(G))$ be a (n, m)-graph together with an edge k-labeling $\psi : E(G) \to \{1, 2, \ldots, k\}$. Define a set of vertex weights $W = \{wt(x) : wt(x) = \sum_{xy \in E(G)} \psi(xy)\}$. Edge labeling ψ is to be *modular irregular k-labeling* if there exists a bijective function $\sigma : W \to Z_n$ defined for each weight $wt(x)$ there corresponds an element $y \in Z_n$ such that $wt(x) \equiv y \pmod{n}$. The minimum k for which G has a modular irregular k-labeling is called *modular irregularity strength* of a graph G, denoted by $\mathrm{ms}(G)$. Muthugurupackiam et al. [11] determined the exact value of modular irregularity strength of bistar graph, cartesian product of graphs C_n with K_2 and corona product of path P_n with K_1.

Muthugurupackiam et al. [12] introduced even modular edge irregularity strength of a graph. Let $G = (V, E)$ be a (n, m)-graph together with a vertex k-labeling $\phi : V(G) \to \{1, 2, \ldots, k\}$. Define a set of edge weights $W = \{wt(uv) : wt(uv) = \phi(u)+\phi(v), \forall\, uv \in E\}$ and let $M = \{0, 2, 4, \ldots, 2(m-1)\}$. The vertex labeling ϕ is called an *even modular edge irregular labeling* if there exists a bijective map $\sigma : W \to M$ defined by $\sigma(wt(uv)) = x$ where $wt(uv) \equiv x \pmod{2m}$. The minimum k for which the graph G has an even modular edge irregular k-labeling is called *even modular edge irregularity strength* of the graph G, and denoted by emes(G). If there doesn't exist an even modular edge irregular labeling for G, then emes$(G) = \infty$. Further more they determined *emes* value of cycle, path, star graph, rectangular graph, disjoint union of paths and disjoint union of cycles.

Motivated by the even modular edge irregularity strength of graphs we introduce a new parameter, an odd modular edge irregularity strength of graph, a modular version of edge irregularity strength. The main aim of this paper is to show a lower bound of the parameter odd modular edge irregularity strength and determine the precise values of this parameter for some families of graphs.

Let $G = (V, E)$ be a (n, m)-graph together with a vertex k-labeling $\rho : V \to \{1, 2, \ldots, k\}$. Define a set of edge weight $W = \{wt(uv) : wt(uv) = \rho(u) + \rho(v), \forall\ uv \in E\}$ and let $M = \{1, 3, 5, \ldots, 2m-1\}$. The vertex k-labeling ρ is called an *odd modular edge irregular labeling* if there exists a bijective map $\sigma : W \to M$ defined by $\sigma(wt(uv)) = x$ where $wt(uv) \equiv x \pmod{2m}$. The minimum k for which the graph G has an odd modular edge irregular k-labeling is called *odd modular edge irregularity strength* of the graph G, and denoted by omes(G). If there doesn't exist an odd modular edge irregular labeling for G, we define $omes(G) = \infty$.

2 Main Results

Following theorem gives the lower bound of odd modular edge irregularity strength of a graph.

Theorem 2.1. Let G be a (n,m)-graph. Then $omes(G) \geq m + 1$.

Proof: Consider a (n,m)-graph G together with a vertex k-labeling ρ. Assume that W is the set of edge weights of G under ρ. If ρ is an odd modular edge irregular labeling, then there must be an edge $e \in E(G)$ such that $wt(e) \equiv 1 \pmod{2m}$. Since the value of $wt(e)$ must be at least $2m + 1$, $omes(G) \geq m + 1$. ∎

Theorem 2.2. For cycle C_n, $n \geq 4$,

$$omes(C_n) = \begin{cases} n+1, & \text{if } n \equiv 0 \pmod 4, \\ \infty, & \text{otherwise.} \end{cases}$$

Proof: Let $V(C_n) = \{v_i : i = 1, 2, \ldots, n\}$ be the vertex set and let $E(C_n) = \{e_i = v_i v_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{e_n = v_n v_1\}$ be the edge set of the cycle C_n.

Case(i). When $n \equiv 0 \pmod 4$ define the vertex labeling $\rho : V(C_n) \to \{1, 2, \ldots, n+1\}$ as follows:

For $1 \leq i \leq \frac{n}{2} + 1$,

$$\rho(v_i) = \begin{cases} 2i-1, & \text{i is odd} \\ 2i-2, & \text{i is even} \end{cases}$$

For $1 \leq i \leq \frac{n}{2} - 1$,

$$\rho(v_{n+1-i}) = \begin{cases} 2i+2, & \text{i is odd} \\ 2i+1, & \text{i is even} \end{cases}$$

The set of edge weights under ρ is $W = \{3, 5, \ldots, 2m+1\}$. Existence of bijective map $\sigma : W \to \{1, 3, \ldots 2m+1\}$ is trivial, which is defined by $\sigma(w) = x$, where $w \equiv x \pmod{2m}$. Hence ρ is an odd modular edge irregular labeling of C_n, thus $omes(C_n) \leq n+1$ is obtained. Lower bound $omes(C_n) \geq n+1$ can be obtained from Theorem 1.

Case (ii). Let $n \equiv 2 \pmod 4$. Assume that C_n has odd modular edge irregular labeling ρ and let W be the set of edge weights of G under ρ, then there exists a bijective mapping $\sigma : W \to \{1, 3, \ldots 2m-1\}$, defined by $\sigma(w) = x$, such that $w \equiv x \pmod{2m}$. Since $n \equiv 2 \pmod 4$, $\sum\limits_{w \in W} w \equiv 0 \pmod 4$. Furthermore, sum of the vertex labels is equal to half of the sum of the weights, i.e $\frac{\sum\limits_{w \in W} w}{2} = \sum\limits_{u \in V} \rho(u)$. To obtain odd edge weights, odd and even labels must be assign alternatively to the vertices of C_n. Thus the number of odd labels will be odd and hence its sum is odd. Hence, $\frac{\sum\limits_{w \in W} w}{2}$ can not be equal to $\sum\limits_{u \in V} \rho(u)$, which is a contradiction. Hence, odd modular edge irregular labeling of C_n does not exist when $n \equiv 2 \pmod 4$.

Case (iii). Suppose n is odd, alternate assignment of odd and even labels for all the vertices of C_n are not possible. Thus, odd modular edge irregular labeling does not exist. Hence the theorem. ∎

Corollary 2.3. Let G be any graph which contains a cycle of order $n \not\equiv 0 \pmod 4$, then $omes(G) = \infty$.

Disjoint union of t copies of path P_n is denoted as tP_n. $omes$ value of tP_n is determined in the following theorem.

Theorem 2.4. For $n \geq 2$ and $t \geq 2$, then $omes(tP_n) = nt - t + 1$.

Proof: Consider the vertex set and edge set of tP_n as $V(tP_n) = \{u_{i,j} : 1 \leq i \leq t, 1 \leq j \leq n\}$ and $E(tP_n) = \{u_{i,j}u_{i,j+1} : 1 \leq i \leq t, 1 \leq j \leq n-1\}$ respectively .

Define the vertex labeling $\rho : V(tP_n) \to \{1, 2, \ldots, nt - t + 1\}$ as follows:

$$\rho(u_{i,j}) = ni - n - i + j + 1, \quad 1 \leq i \leq t, \quad 1 \leq j \leq n.$$

Vertex labeling ρ induce the set of edge weights $W = \{3, 5, \ldots, 2(n-1)t+1\}$. Existence of bijective map $\sigma : W \to \{1, 3, \ldots, 2m-1\}$, defined by $\sigma(w) = x$, such that $w \equiv x \pmod{2m}$ is trivial, where m denotes the number of edges of tP_n. Thus ρ is an odd modular edge irregular labeling of tP_n, and upper bound

$$omes(tP_n) \leq nt - t + 1$$

is obtained.
Lower bound $omes(tP_n) \geq nt - t + 1$ can be obtained directly from Theorem 1. Hence the theorem. ∎

Observe that the lower bound given in Theorem 1 is tight for the graph tP_n. But for the star graph odd modular edge irregularity strength is high compared to the lower bound given in Theorem 1. In the next theorem we present odd modular edge irregularity strength of star graph.

Theorem 2.5. Let $K_{1,n}$ be a star graph of order $n+1, n \geq 1$. Then

$$omes(K_{1,n}) = 2n - 1.$$

Proof: Let $V(K_{1,n}) = \{u, v_i : i = 1, 2, \dots, n\}$ be the vertex set and let $E(K_{1,n}) = \{e_i = uv_i : i = 1, 2, \dots, n\}$ be the edge set of the star $K_{1,n}$.

Define the vertex labeling $\rho : V(K_{1,n}) \to \{1, 2, \dots, 2n-1\}$ as follows:

$$\rho(u) = 2,$$
$$\rho(v_i) = 2i - 1, 1 \le i \le n.$$

The set of edge weights under the labeling ρ is $W = \{3, 5, \dots, 2n+1\}$. Existence of a bijective map $\sigma : W \to M$ defined by $\sigma(w) = x$ such that $w \equiv x \pmod{2n}$, (where $M = \{1, 3, 5, \dots, 2n-1\}$) is trivial. Thus ρ is an odd modular edge irregular labeling of $K_{1,n}$, and $omes(K_{1,n}) \le 2n - 1$.
Consider the set of least possible values of distinct odd edge weights $W(K_{1,n}) = \{3, 5, \dots, 2n+1\}$. To obtain the weight 3, the vertex u must be label with 1 or 2, at the same time the heaviest weight $2n+1$ can be obtained by assigning label at least $2n$ or $2n-1$ to any v_i. Thus, $omes(K_{1,n}) \ge 2n - 1$. Hence the theorem. ■

Binary rooted tree T_n is a tree in which there is exactly one vertex (root) of degree two, and each of the remaining vertices is of degree one or three. Clearly the number of vertices n in such a tree is always odd. A vertex v_i is said to be at level l_i if v_i is at a distance of l_i from the root. The maximum level l_{max} of any vertex in a binary tree is called the height of the tree.

Theorem 2.6. For odd $n \ge 3$, let T_n be a binary rooted tree of order n whose height is $\frac{n-1}{2}$. Then $omes(T_n) = n$.

Proof: Let $V(T_n) = \{v_i : i = 0, 1, 2, \dots, n-1\}$ be the vertex set and let $E(T_n) = \{v_{2i}v_{2i+1}, v_{2i}v_{2i+2} : i = 0, 1, 2, \dots, \frac{n-1}{2}\}$ be the edge set of the binary rooted tree T_n.

Define the vertex labeling $\rho : V(T_n) \to \{1, 2, \dots, n\}$ as follows:
Case(i): Suppose $\frac{n-1}{2}$ is odd.
For $0 \le i \le \frac{n-1}{2}$,

$$\rho(v_{2i}) = \begin{cases} 2i+1, & \text{if i is odd} \\ 2i+2, & \text{if i is even.} \end{cases}$$

For $1 \leq i \leq \frac{n-1}{2}$,

$$\rho(v_{2i-1}) = \begin{cases} 2i-1, & \text{if i is odd} \\ 2i, & \text{if i is even.} \end{cases}$$

Case(ii): Suppose $\frac{n-1}{2}$ is even.

For $0 \leq i \leq \frac{n-1}{2}$,

$$\rho(v_{2i}) = \begin{cases} 2i+2, & \text{if i is odd} \\ 2i+1, & \text{if i is even.} \end{cases}$$

For $1 \leq i \leq \frac{n-1}{2}$,

$$\rho(v_{2i-1}) = \begin{cases} 2i, & \text{if i is odd} \\ 2i-1, & \text{if i is even.} \end{cases}$$

The set of edge weights $W = \{3, 5, ..., 2n-1\}$. Existence of the bijective map $\sigma : W \rightarrow \{1, 3, ..., 2n-3\}$ is trivial and hence the upper bound $omes(T_n) \leq n$ is obtained. The lower bound $omes(T_n) \geq n$ can be obtained directly from Theorem 1. ∎

Ladder graph $L_n = K_2 \times P_n, n \geq 3$ is formed by taking two copies of P_n and joining the corresponding vertices by an edge. Let $V = \{u_i, v_i : 1 \leq i \leq n\}$ be the vertex set and let $E = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\}$ be the edge set of L_n. The following theorem gives the precise value of odd modular edge irregularity strength of ladder graph.

Theorem 2.7. Let L_n be a ladder graph of order $2n$, $n \geq 3$. Then $omes(L_n) = 3n - 1$.

Proof: Define the vertex labeling $\rho : V \rightarrow \{1, 2, \ldots, 3n-1\}$ as follows:

$$\rho(v_i) = 3i - 2, \quad 1 \leq i \leq n,$$
$$\rho(u_i) = 3i - 1, \quad 1 \leq i \leq n.$$

Clearly, ρ is an odd modular edge irregular labeling of L_n and hence upper bound $omes(L_n) \leq 3n - 1$ is obtained. By Theorem 1, $omes(L_n) \geq 3n - 1$. ∎

Square grid graph $S_n = P_n \times P_n, n \geq 3$ is formed by taking cartesian product of P_n with P_n. Let $V = \{u_{i,j} : 1 \leq i, j \leq n\}$ be the vertex set and let $E = \{u_{i,j} u_{i,j+1} : 1 \leq$

$i \leq n, 1 \leq j \leq n-1\} \cup \{u_{i,j}u_{i+1,j} : 1 \leq i \leq n-1, 1 \leq j \leq n\}$ be the edge set of S_n. The following theorem gives the precise value of odd modular edge irregularity strength of square grid graph.

Theorem 2.8. Let S_n be a square grid graph of order n^2, $n \geq 2$. Then $omes(S_n) = 2n^2 - 2n + 1$.

Proof: Define the vertex labeling $\rho : V \to \{1, 2, \ldots, 2n^2 - 2n + 1\}$ as follows:

$$\rho(u_{i,i}) = 4n(i-1) - 2i(i-1) + 1, \quad 1 \leq i \leq n.$$

When $i < j$,

$$\rho(u_{i,j}) = \begin{cases} \rho(u_{i,i}) + 2(j-i) - 1, & \text{if } j - i \text{ is odd} \\ \rho(u_{i,i}) + 2(j-i), & \text{if } j - i \text{ is even.} \end{cases}$$

When $i > j$,

$$\rho(u_{i,j}) = \begin{cases} \rho(u_{j,j}) + 2(i-j) + 1, & \text{if } i - j \text{ is odd} \\ \rho(u_{j,j}) + 2(i-j), & \text{if } i - j \text{ is even.} \end{cases}$$

From the above odd modular edge irregular labeling ρ upper bound $omes(S_n) \leq 2n^2 - 2n+1$ is obtained. Lower bound can be obtained from Theorem 1. Hence the theorem. ∎

3 Conclusion

In this paper, the study of a new modular version of edge irregularity strength of a graph called odd modular edge irregularity strength was introduced and estimated it's value for chosen families of graphs. There are many graphs for which odd modular edge irregularity strength to be determined. We left it open for the readers. In particular, we present two following open problems.
Open Problem 3.1. Determine $omes(T_n)$, when height of $T_n \neq \frac{n-1}{2}$.
Open Problem 3.2. Determine $omes(P_m \times P_n)$, when $m \neq n$.

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